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CHINA AGRICULTURAL UNIVERSITY

能源与动力工程专业英语（2024秋）

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第二章 笛卡尔张量
Chapter 2 Cartesian Tensors

October 29, 2024

2.1 Scalars, Vectors, Tensors, Notation

The physical quantities in fluid mechanics vary in their complexity, and may involve multiple spatial directions.

Scalars or **zero-order** tensors may be defined with a single magnitude and appropriate units, may vary with spatial location, but are independent of coordinate directions.

Scalars are typically denoted herein by italicized symbols. For example, common scalars in fluid mechanics are pressure p, temperature T, and density ρ.

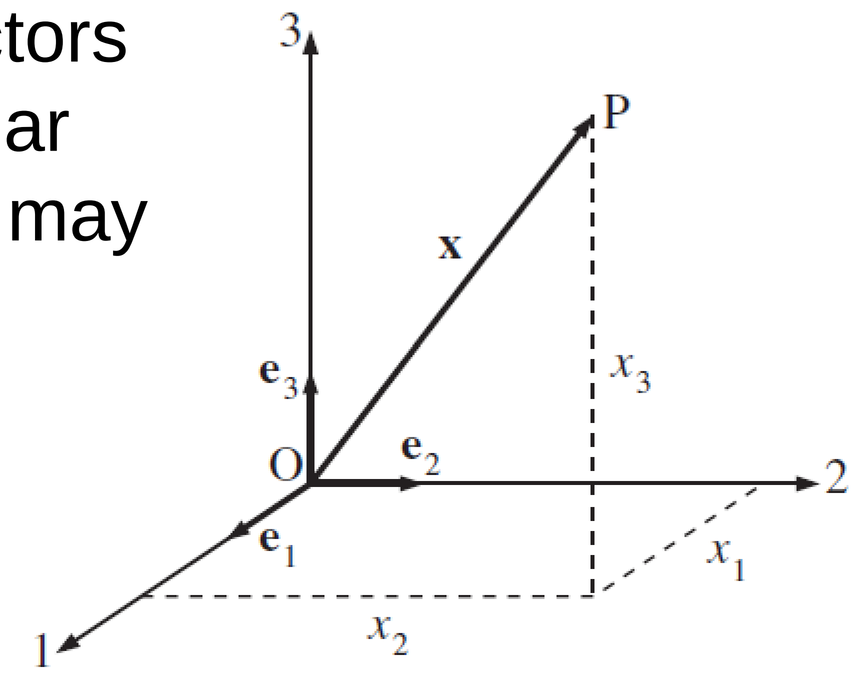
2.1 Scalars, Vectors, Tensors, Notation

Vectors or **first-order tensors** have both a magnitude and a direction. A vector can be completely described by its components along three orthogonal coordinate directions.

A vector is usually denoted herein by a boldface symbol. For example, common vectors in fluid mechanics are position $\mathbf{x}$, fluid velocity $\mathbf{u}$, and gravitational acceleration $\mathbf{g}$.

2.1 Scalars, Vectors, Tensors, Notation

In a Cartesian coordinate system with unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, in the three mutually perpendicular directions, the position vector $\mathbf{x}$, OP in Figure, may be written



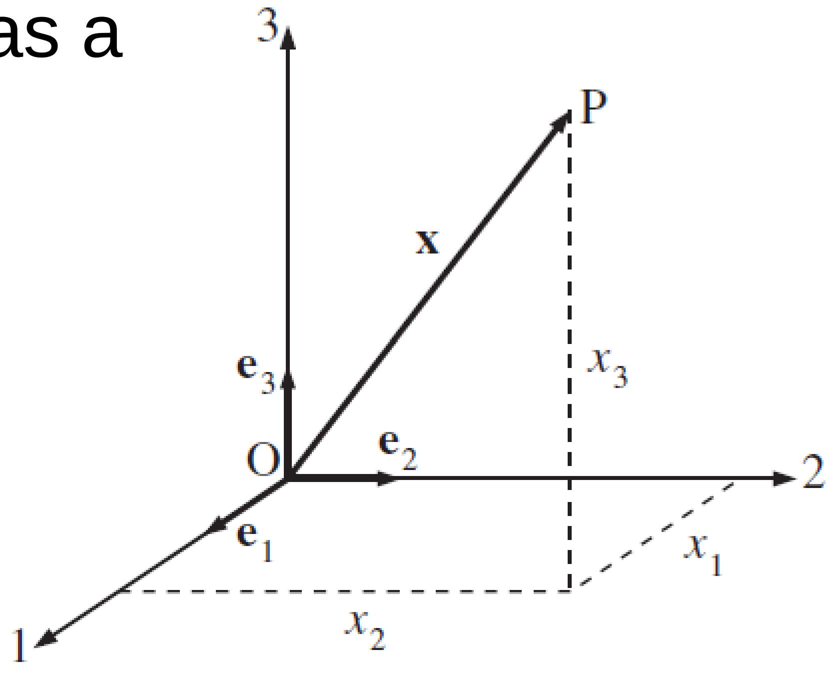
$$\mathbf{x} = \mathbf{e}_1 x_1 + \mathbf{e}_2 x_2 + \mathbf{e}_3 x_3$$

The components of $\mathbf{x}$ can be denoted by x_i or x_j (or x_k, etc.).

2.1 Scalars, Vectors, Tensors, Notation

For algebraic manipulation, a vector is written as a column matrix

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{where} \quad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



The **transpose** of the matrix (denoted by a superscript T) is obtained by interchanging rows and columns, so the transpose of the column matrix x is the row matrix:

$$\mathbf{x}^{\mathrm{T}} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

2.1 Scalars, Vectors, Tensors, Notation

Second-order tensors have a component for each pair of coordinate directions and therefore may have as many as 3×3=9 separate components.

A second-order tensor is sometimes denoted by a boldface symbol. For example, a common second-order tensor in fluid mechanics is the stress $\boldsymbol{\tau}$.

Once a coordinate system is chosen, the nine components of a second-order tensor can be represented by a 3×3 matrix, or by an italic symbol having two indices, such as τ_{ij} for the stress tensor.

2.2 Einstein Summation Convention 爱因斯坦求和约定

This notational convention can be stated as follows:

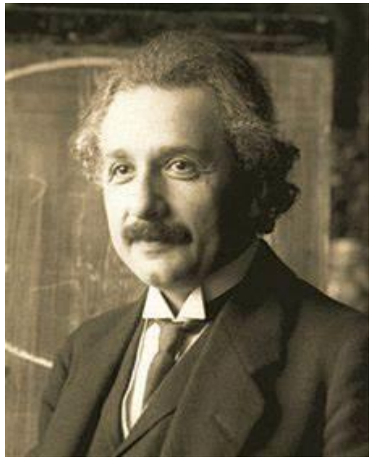
Whenever an index is repeated in a term, a summation over this index is implied, even though no summation sign is explicitly written.

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i \equiv a_i b_i,$$

Similarly, the state of stress at a point can be completely specified by the nine components τ_{ij} (where $i, j = 1, 2, 3$) that can be written as the matrix

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

2.2 Einstein Summation Convention 爱因斯坦求和约定



Albert Einstein

父-子

Hans Albert Einstein

师-生

钱宁先生

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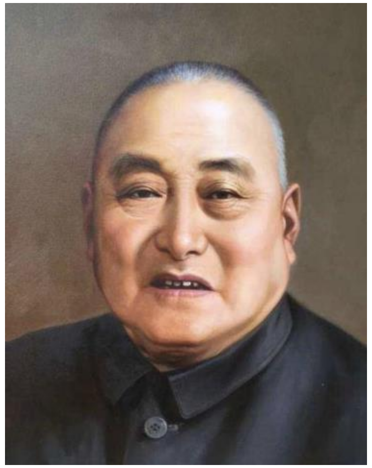
2.2 Einstein Summation Convention 爱因斯坦求和约定

徐特立

生-师



黄炎培

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2.3 Vector, Dot, and Cross Products

The ***dot product*** of two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as the scalar

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i \equiv u_i v_i$$

It is easy to show that $\mathbf{u}\cdot\mathbf{v} = \cos\theta$, where u and v are the vectors' magnitudes and θ is the angle between the vectors.

The dot product is therefore the magnitude of one vector times the component of the other in the direction of the first. The dot product $\mathbf{u}\cdot\mathbf{v}$ is equal to the sum of the diagonal terms of the tensor $u_i v_j$.

2.3 Vector, Dot, and Cross Products

The ***cross product*** between two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as the vector $\mathbf{w}$ whose magnitude is $uv \sin\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and whose direction is perpendicular to the plane of $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form a right-handed system.

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{e}_1 + (u_3 v_1 - u_1 v_3)\mathbf{e}_2 + (u_1 v_2 - u_2 v_1)\mathbf{e}_3$$

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

In indicial notation, the k-component of $\mathbf{u} \times \mathbf{v}$ can be written as

$$(\mathbf{u} \times \mathbf{v})_k = \sum_{i=1}^{3} \sum_{j=1}^{3} \varepsilon_{ijk} u_i v_j \equiv \varepsilon_{ijk} u_i v_j = \varepsilon_{kij} u_i v_j$$

2.3 Vector, Dot, and Cross Products

It is called ***the alternating tensor*** or ***permutation symbol***, and is defined as

$$\varepsilon_{ijk} = \begin{Bmatrix} 1 & \text{if } ijk = 123, 231, \text{ or } 312 \text{ (cyclic order)}, \\ 0 & \text{if any two indices are equal}, \\ -1 & \text{if } ijk = 321, 213, \text{ or } 132 \text{ (anti-cyclic order)} \end{Bmatrix}$$

2.4 Gradient, Divergence, and Curl

The vector-differentiation operator is defined symbolically by

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} = \sum_{i=1}^{3} \mathbf{e}_i \frac{\partial}{\partial x_i} \equiv \mathbf{e}_i \frac{\partial}{\partial x_i}$$

The inverted Greek delta is called a “nabla”

When operating on a scalar function of position ϕ , it generates the vector

$$\nabla \phi = \sum_{i=1}^{3} \mathbf{e}_i \frac{\partial \phi}{\partial x_i} \equiv \mathbf{e}_i \frac{\partial \phi}{\partial x_i}$$

The vector $\nabla\phi$ is called the ***gradient*** of ϕ, and $\nabla\phi$ is perpendicular to surfaces defined by $\phi = \text{constant}$

2.4 Gradient, Divergence, and Curl

The ***divergence*** of a vector field $\mathbf{u}$ is defined as the scalar

$$\nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} \equiv \frac{\partial u_i}{\partial x_i}$$

The ***curl*** of a vector field $\mathbf{u}$ is defined as $\nabla \times \mathbf{u}$, whose i-component can be written as

$$(\nabla \times \mathbf{u})_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \equiv \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

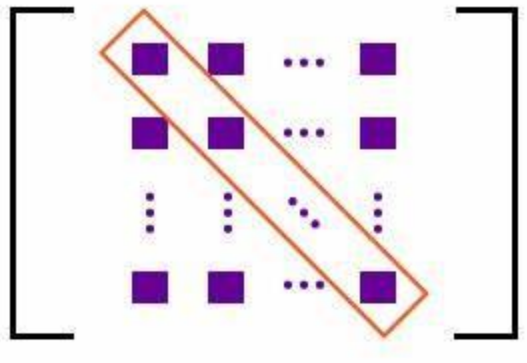
2.4 Gradient, Divergence, and Curl

A vector field $\mathbf{u}$ is called ***divergence free*** if $\nabla \cdot \mathbf{u} = \mathbf{0}$, and ***irrotational*** or ***curl free*** if $\nabla \times \mathbf{u} = \mathbf{0}$.

2.5 Symmetric and Antisymmetric Tensors

A tensor $\mathbf{B}$ is called **symmetric** in the indices i and j if the components do not change when i and j are interchanged, that is, if $B_{ij} = B_{ji}$.

Thus, the matrix of a symmetric second-order tensor is made up of only six distinct components (the three on the diagonal where $i = j$, and the three above or below the **diagonal** where $i \neq j$) .



2.5 Symmetric and Antisymmetric Tensors

On the other hand, a tensor is called **antisymmetric** if $B_{ij} = -B_{ji}$.

An antisymmetric tensor must have zero diagonal components, and is made up of only three distinct components.

2.5 Symmetric and Antisymmetric Tensors

Any tensor can be represented as the sum of a symmetric part and an antisymmetric part. For if we write

antisymmetric part

$$B_{ij} = \frac{1}{2}(B_{ij} + B_{ji}) + \frac{1}{2}(B_{ij} - B_{ji}) = S_{ij} + A_{ij}$$

symmetric part

then the operation of interchanging i and j does not change the first term, but changes the sign of the second term.

2.5 Symmetric and Antisymmetric Tensors

Every vector can be associated with an antisymmetric tensor, and vice versa.

For example, we can associate the vector $\boldsymbol{\omega}$ having components ω_i, with an antisymmetric tensor:

$$\mathbf{R} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

The two are related via

$$R_{ij} = \sum_{k=1}^{3} -\varepsilon_{ijk}\omega_k \equiv -\,\varepsilon_{ijk}\omega_k, \quad \text{and} \quad \omega_k = \sum_{i-1}^{3}\sum_{j=1}^{3} -\frac{1}{2}\varepsilon_{ijk}R_{ij} \equiv -\frac{1}{2}\varepsilon_{ijk}R_{ij}$$

2.6 Eigenvalues and Eigenvectors of a Symmetric Tensor

Suppose $\boldsymbol{\tau}$ is a symmetric tensor with real elements, for example, the stress tensor. Then the following facts can be proved:

(1) There are three real **eigenvalues** $\lambda^k (k = 1,2,3)$, which may or may not all be distinct. (Here, the superscript k is not an exponent, and λ^k does not denote the k-component of a vector.)

These eigenvalues (λ^1, λ^2, and λ^3) are the roots or solutions of the **third-degree polynomial**

$$\det \left| \tau_{ij} - \lambda \delta_{ij} \right| = 0$$

2.6 Eigenvalues and Eigenvectors of a Symmetric Tensor

(2) The three **eigenvectors** $\mathbf{b}^k$ corresponding to distinct eigenvalues λ^k are mutually orthogonal.

These eigenvectors define the directions of the **principal axes** of $\boldsymbol{\tau}$. Each $\mathbf{b}$ is found by solving three algebraic equations,

$$\left(\tau_{ij} - \lambda\delta_{ij}\right)b_j = 0$$

2.6 Eigenvalues and Eigenvectors of a Symmetric Tensor

(3) If the coordinate system is rotated so that its unit vectors coincide with the eigenvectors, then $\boldsymbol{\tau}$ is diagonal with elements λ^k in this rotated coordinate system,

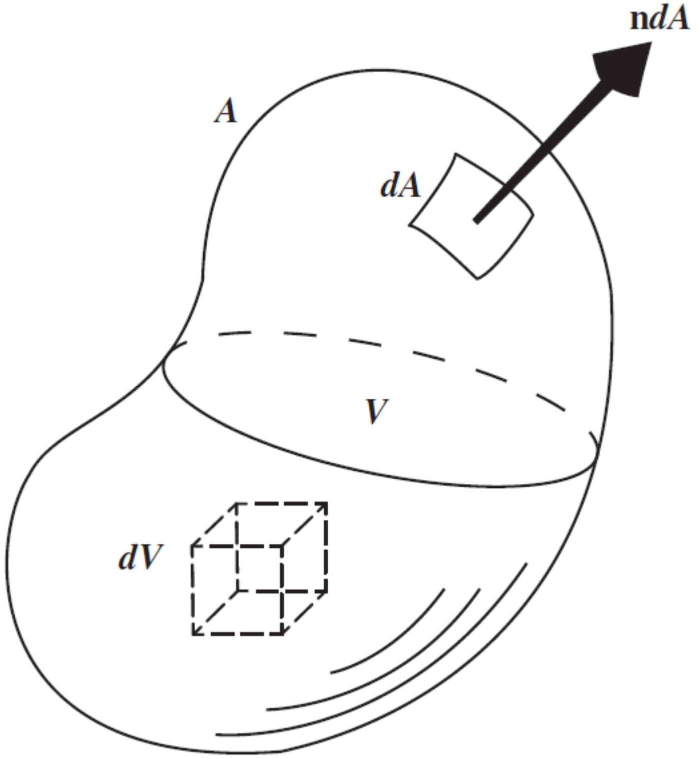
$$\tau' = \begin{bmatrix} \lambda^1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^3 \end{bmatrix}$$

(4) Although the elements τ_{ij} change as the coordinate system is rotated, they cannot be larger than the largest λ or smaller than the smallest λ; the λ^k represent the extreme values of τ_{ij}.

2.7 Gauss' Theorem

This very useful theorem relates volume and surface integrals. Let V be a volume bounded by a closed surface A.

Consider an infinitesimal (无穷小) surface element dA having outward unit normal $\mathbf{n}$ with components n_i, and let $Q(\mathbf{x})$ be a scalar, vector, or tensor field of any order.



Gauss' theorem states that

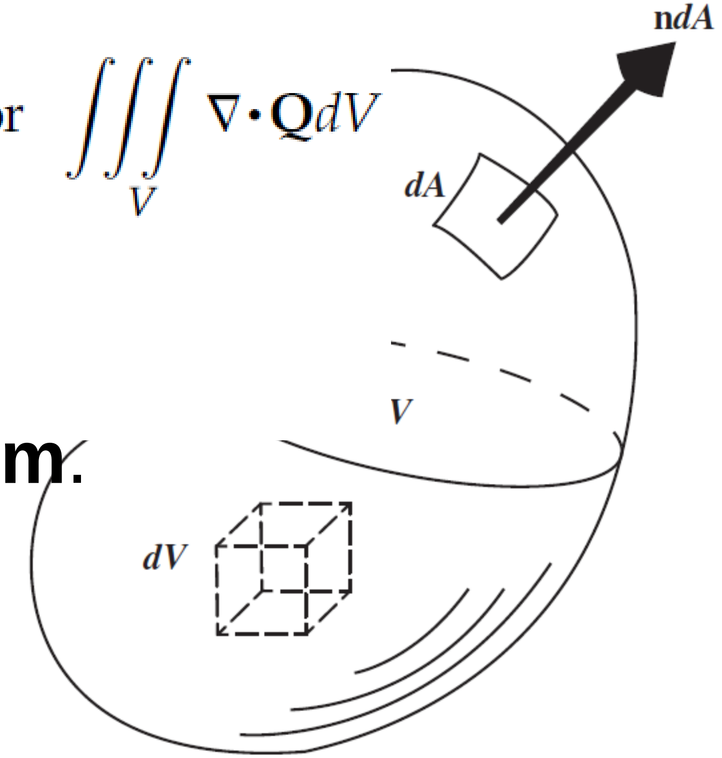
$$\iiint_V \frac{\partial Q}{\partial x_i} dV = \iint_A n_i Q dA$$

2.7 Gauss' Theorem

The most common form of Gauss' theorem is when **Q** is a vector, in which case the theorem is

$$\iiint_V \sum_{i=1}^{3} \frac{\partial Q_i}{\partial x_i} dV \equiv \iiint_V \frac{\partial Q_i}{\partial x_i} dV = \iint_A \sum_{i=1}^{3} n_i Q_i \, dA \equiv \iint_A n_i Q_i \, dA, \quad \text{or} \quad \iiint_V \nabla \cdot \mathbf{Q} dV$$

$$= \iint_A \mathbf{n} \cdot \mathbf{Q} dA,$$



which is commonly called the **divergence theorem**.

In words, the theorem states that the volume integral of the divergence of **Q** is equal to the surface integral of the outflux of **Q**.

END

2.1 Scalars, Vectors, Tensors, Notation

2.2 Einstein Summation Convention 爱因斯坦求和约定

2.3 Vector, Dot, and Cross Products

2.4 Gradient, Divergence, and Curl

ANY QUESTIONS?

2.5 Symmetric and Antisymmetric Tensors

2.6 Eigenvalues and Eigenvectors of a Symmetric Tensor

2.7 Gauss' Theorem